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WEIGHTED REVERSE FRACTIONAL INEQUALITIES OF MINKOWSKI'S AND HÖLDER'S TYPE

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Abstract. General weighted fractional integral operator is introduced by means of some weighted classes. This operator reduces to many well known fractional integral operators. Some weighted Minkowski's reverse fractional integral inequalities, weighted Hölder's reverse fractional integral inequalities and weighted integral inequalities of arithmetic and geometric means are established. At the end, some applications and examples are given.

Keywords: weighted integral inequalities, weighted fractional operator, Minkowski's integral inequalities, Hölder integral inequalities.

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1. Introduction

In the last years, many specialists of several fields have found different results about some well-know inequalities and applications by means of the generalization of the Riemann-Liouville fractional derivative, Riemann-Liouville fractional integral operator, Saigo fractional integral operator, Hadamard integral operator and some other, see [1, 5, 8, 12, 14, 16]. Recently, it has grew up the interest to get new results and interesting relations about fractional integral inequalities using the above operators. In this paper, we integrate all these operators and give a general results by means of weighed classes. Besides, our results reduce to many well known integral inequalities for the most simples cases, just considering some suitable weights.

Everywhere below, we assume that λ is said to be of the class Δ, if the function $\lambda : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous with respect to one of their variables in $[0,\infty)$. Now, if $\lambda \in \Delta$ and $f(\tau)$ is a real-valued continuous function given in $[0,\infty)$, we define a weighted operator:

$$I_\lambda[f(t)] = \int_a^t \lambda^{(\alpha,\beta,\kappa)}(\tau,t) f(\tau)d\tau, \quad a \le t \le +\infty, \tag{1}$$

where $a \ge 0$ and the weight λ depends on some complex parameters α, β, κ. This operator is in a sense the same used in [1], but the weighted classes Δ used to evaluate the operator are most general than the class Ω introduced in [1]. Besides, one can prove easily that Ω is a subset of Δ. Hence, the operator introduced in this paper shall arise more applications and results in differential equations, integral inequalities, special functions, fractional calculus, etc. (see [21, 15, 23]).

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Remark 1.1. *Note that the integral operator* (1) *could have as an endpoint* $+\infty$ *of the interval of integration approaches, in this case we shall understand this like an improper integral.*

2. Preliminaries

We recall a definition about the generalized gamma function. After that some facts are established.

Definition 2.1. *Let* $k > 0$, *then the generalized* k*-gamma function defined by* [9]

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \tag{2}$$

where $(x)_{n,k}$ *is the Pochhammer* k*-symbol defined by*

$$(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k) \quad (n \geq 1).$$

Now, we shall present some of the most important and interesting remarks about several applications of our weighted classes and operator I_λ. These remarks show that we can become the results of this paper in many different type of fractional calculus.

Remark 2.1. *If* $\lambda^{c,\alpha,\eta}(\tau,t) = t^\eta\left(1-\frac{\tau}{t}\right)^{\frac{\eta}{1-\alpha}}$ *where* $\eta \in \mathbb{C}$, $\operatorname{Re}\eta > 0$, $c > 0$ *and* $\alpha < 1$, *then* $I_\lambda[f\left(\frac{t}{c(1-\alpha)}\right)]$ *becomes to the pathway fractional integral operator in* [18], *for* $a = 0$ *and* $f(t) \in L(c,b)$.

Remark 2.2. *If* $\lambda(t,\tau) = \frac{1}{\tau}\left(\log\frac{t}{\tau}\right)^{\alpha-1}$ *where* $\alpha > 0$ *and* $t \geq \tau \in [a,b]$ $(a \geq 1)$, *then* I_λ *becomes to the classical left-sided Hadamard integral of fractional order* α *in* [17], *i.e.*

$$I_\lambda[f(t)] = \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{f(\tau)}{\tau}d\tau, \quad t \in [a,b].$$

Remark 2.3. *If* $\lambda(t,\tau) = \frac{[h(t)-h(\tau)]^{\alpha-1}h'(\tau)}{\Gamma(\alpha)}$ *with* $\alpha > 0$ *and* $\tau \in (a,t)$, $h(\tau)$ *is an increasing and a positive monotone function on* $(a,b]$, *having a continuous derivative* $h'(\tau)$ *on* (a,b), *then* $I_\lambda[f(t)]$ *becomes to* $J^\alpha_{a^+,h}f$ *in* [14].

Remark 2.4. *If* $\lambda^{(\alpha,k,r)}(t,\tau) = \frac{(1+r)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}\tau^r$ *for* $a \leq \tau \leq t$, $k > 0$ *and* $r \in \mathbb{R}\setminus\{-1\}$, *we get the generalized Riemann-Liouville* k*-fractional integral* $R^{\alpha,r}_{a,k}$ *of order* $\alpha > 0$ *introduced in* [23], *i.e.* $I_\lambda[f(t)] = R^{\alpha,r}_{a,k}\{f(t)\}$. *Besides, this definition coincide with the* $(k;r)$*-Riemann-Liouville fractional integral of* f *of order* $\alpha > 0$ *in* [23]. *Moreover, setting* $r = 0$, $I_\lambda[f(t)]$ *is the Riemann-Liouville* k*-fractional integral defined in* [24].

Remark 2.5. *If* $\lambda(t,\tau) = \frac{t^{-\eta-\alpha}}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}\tau^{-\eta}$ *where* $\alpha > 0$ *and* η *is a complex parameter, then for* $a = 0$

$$I_\lambda[f(t)] = \frac{t^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{-\eta}f(\tau)d\tau,$$

is the Erdélyi-Kober fractional integral of [16, 10] *which generalizes the Riemann fractional integral and the Weyl integral (see* [19]*).*

Remark 2.6. *If* $\lambda^{\alpha,\rho}(t,\tau) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}$ *where* $\operatorname{Re}\alpha > 0$ *and* $\rho \in \mathbb{R} \neq \{-1\}$, *then the operator*

$$I_\lambda[f(t)] = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}f(\tau)d\tau = ({}^\rho I^\alpha_{a+}f)(t), \quad t > a,$$

is called the left-sided Katugampola fractional integral (see [11, 12]*). Analogously, it is defined right-sided fractional integral with a little bit changes.*

Remark 2.7. *If* $\lambda^{(x,y,k)}(t,\tau) = \frac{\tau^{\frac{x}{k}-1}(1-\tau)^{\frac{y}{k}-1}}{kf(\tau)}$ *for* $t \geq \tau \geq 0$*,* $Re\ x > 0$*,* $Re\ y > 0$*,* $k > 0$ *and* f *is a positive and continuous function on* $[0,1]$*, then*

$$I_\lambda[f(t)] = \frac{1}{k}\int\limits_0^t \tau^{\frac{x}{k}-1}(1-\tau)^{\frac{y}{k}-1}d\tau = \beta_k^{[0,t]}(x,y).$$

And, $\beta_k^{[0,t]}(x,y)$ *becomes to the* k*-beta function in* [9] *when* $t = 1$*. Besides, if* $\lambda(t,\tau) = \frac{\tau^{x-1}(1-\tau)^{y-1}}{k}$*, then*

$$I_\lambda[1(t)] = \beta_k^{[0,t]}(x,y).$$

Remark 2.8. *If* $\lambda^{(\alpha,\beta,\eta)}(\tau,t) = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$ *where* $\alpha > 0$*,* $t \geq \tau \geq 0$ *and* $\beta,\eta \in \mathbb{C}\setminus\mathbb{Z}^-$*, then the operator* $I_\lambda[f(t)]$ *becomes to the Saigo generalized fractional integral* $I_{0,x}^{\alpha,\beta,\eta}[f(t)]$ *(see* [22]*).*

Remark 2.9. *If* $\lambda^\alpha(\tau,t) = \tau^{1-\alpha}$ *for* $\alpha \in (0,1)$*, then* $I_\lambda[f(t)] = I_\alpha^a(f)(t)$*, i.e. the conformal fractional integral defined in* [13].

3. Weighted Minkowski's reverse fractional integral inequalities

In this section we prove some theorems on Minkowski's reverse fractional integral inequality.

Theorem 3.1. *Let* $p \geq 1$*,* $\lambda \in \Delta$ *and let* f*,* g *be two positive functions on* $[0,+\infty)$ *such that for all* $t > 0$*,* $I_\lambda[f^p(t)] < \infty$*,* $I_\lambda[g^p(t)] < \infty$*. If* $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$*,* $\tau \in [a,t]$ $(a \geq 0)$*, then*

$$\left(I_\lambda[f^p(t)]\right)^{1/p} + (I_\lambda[g^p(t)])^{1/p} \leq \frac{1+M(m+2)}{(m+1)(M+1)}(I_\lambda[(f+g)^p(t)])^{1/p}. \tag{3}$$

Proof. By the condition $\frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [a,t]$ $(t > a)$, it follows

$$(M+1)^p f^p(\tau) \leq M^p(f+g)^p(\tau). \tag{4}$$

Multiplying both sides of (4) by $\lambda(\tau,t)$ and integrating respect to τ over (a,t), we get

$$(M+1)^p\int\limits_a^t \lambda(\tau,t)f^p(\tau)d\tau \leq M^p\int\limits_a^t \lambda(\tau,t)(f+g)^p(\tau)d\tau,$$

This imply,

$$(I_\lambda[f^p(t)])^{1/p} \leq \frac{M}{M+1}I_\lambda[(f+g)^p(t)])^{1/p}. \tag{5}$$

Besides, by the condition $m \leq \frac{f(\tau)}{g(\tau)}$, we obtain

$$\left(1+\frac{1}{m}\right)g(\tau) \leq \frac{1}{m}(f(\tau)+g(\tau)).$$

Thus,

$$\left(1+\frac{1}{m}\right)^p g^p(\tau) \leq \frac{1}{m^p}(f(\tau)+g(\tau))^p. \tag{6}$$

Hence, multiplying both sides of (6) by $\lambda(\tau,t)$ and integrating respect to τ over (a,t), we get

$$(I_\lambda[g^p(t)])^{1/p} \leq \frac{1}{m+1}(I_\lambda[(f+g)^p(t)])^{1/p}. \tag{7}$$

By (4) and (7), we get the desired result (3). □

Remark 3.1. *For the most simple case, taking $\lambda \equiv 1$, Theorem 3.1 becomes to* [3, Theorem 1.2] *on* $[0,t]$. *Besides, if $\lambda^{\alpha}(\tau,t) = (t-\tau)^{\alpha-1}$, for $\alpha > 0$ and $t > 0$, Theorem 3.1 becomes to* [8, Theorem 2.1] *on* $(0,t)$.

Theorem 3.2. *Let $p \geq 1$, $\lambda \in \Delta$ and let f, g be two positive functions on $[0,+\infty)$ such that for all $t > 0$, $I_\lambda[f^p(t)] < \infty$, $I_\lambda[g^p(t)] < \infty$. If $0 < c < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [a,t]$ $(a \geq 0)$, then*

$$\frac{M+1}{M-c}(I_\lambda[(f-cg)^p(t)])^{1/p} \leq (I_\lambda[f^p(t)])^{1/p} + (I_\lambda[g^p(t)])^{1/p} \leq \frac{m+1}{m-c}(I_\lambda[(f-cg)^p(t)])^{1/p}. \quad (8)$$

Proof. By hypothesis, we get

$$m-c \leq \frac{f(\tau)}{g(\tau)} - c \leq M-c, \quad \tau \in [a,t],\, a \geq 0,$$

or what this the same

$$\frac{f(\tau)-cg(\tau)}{M-c} \leq g(\tau) \leq \frac{f(\tau)-cg(\tau)}{m-c}.$$

Hence, multiplying by $\lambda(\tau,t)$ and integrating respect τ over (a,t) in the last inequality, we get

$$\frac{1}{M-c}\left(\int_a^t \lambda(\tau,t)(f(\tau)-cg(\tau))^p d\tau\right)^{1/p} \leq \left(\int_a^t \lambda(\tau,t)g^p(\tau)d\tau\right)^{1/p}$$
$$\leq \frac{1}{m-c}\left(\int_a^t \lambda(\tau,t)(f(\tau)-cg(\tau))^p d\tau\right)^{1/p}. \quad (9)$$

On the other hand, we have

$$-\frac{1}{m} \leq -\frac{g(\tau)}{f(\tau)} \leq -\frac{1}{M}, \quad \tau \in [a,t].$$

Thus,

$$\frac{1}{c} - \frac{1}{m} \leq \frac{1}{c} - \frac{g(\tau)}{f(\tau)} \leq \frac{1}{c} - \frac{1}{M},$$

i.e.

$$\frac{m-c}{cm} \leq \frac{f(\tau)-cg(\tau)}{cf(\tau)} \leq \frac{M-c}{cM}.$$

Hence,

$$\frac{M}{M-c}(f(\tau)-cg(\tau)) \leq f(\tau) \leq \frac{m}{m-c}(f(\tau)-cg(\tau)).$$

Then, multiplying by $\lambda(\tau,t)$ and integrating respect to τ over (a,t), we obtain

$$\frac{M}{M-c}\left(\int_a^t \lambda(\tau,t)(f(\tau)-cg(\tau))^p d\tau\right)^{1/p} \leq \left(\int_a^t \lambda(\tau,t)f^p(\tau)d\tau\right)^{1/p}$$
$$\leq \frac{m}{m-c}\left(\int_a^t \lambda(\tau,t)(f(\tau)-cg(\tau))^p d\tau\right)^{1/p}. \quad (10)$$

Finally, by (9) and (10) follow (8). □

Remark 3.2. *If $\lambda \equiv 1$, Theorem 3 becomes to Theorem 2.2 in* [25]. *Moreover, if $c = 1$, then we get an integral inequality presented by Sulaiman in* [26].

4. Weighted Hölder's reverse fractional integral inequality

In what follows, are two results in which we intend to establish the Hölder's reverse fractional integral inequality using the weighted integral operator.

Theorem 4.1. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in \Delta$ and let f, g be two positive functions on $[0, \infty[$, such that for all $t > a$, $I_\lambda[f(t)] < \infty$, $I_\lambda[g(t)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M < \infty$, $\tau \in [a, t]$, then we have the following*

$$[I_\lambda f(t)]^{\frac{1}{p}} \left[I_\lambda g(t)\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} I_\lambda \left[(f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}}\right]. \tag{11}$$

Proof. Since $\frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [a, t]$, $a \geq 0$, we have

$$[g(\tau)]^{\frac{1}{q}} \geq M^{\frac{-1}{q}} [f(\tau)]^{\frac{1}{q}} \tag{12}$$

and

$$\begin{aligned} [f(\tau)]^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} &\geq M^{\frac{-1}{q}} [f(\tau)]^{\frac{1}{q}} [f(\tau)]^{\frac{1}{p}} \\ &\geq M^{\frac{-1}{q}} [f(\tau)]^{\frac{1}{q}+\frac{1}{q}} \geq M^{\frac{-1}{q}} [f(\tau)]. \end{aligned} \tag{13}$$

Then, multiplying (13) by $\lambda(\tau, t)$ and integrating respect to τ over (a, t), we obtain

$$I_\lambda \left[[f(t)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}}\right] \geq M^{\frac{-1}{q}} \left[I_\lambda[f(t)]\right]. \tag{14}$$

hence, we can write

$$\left(I_\lambda \left[[f(t)]^{\frac{1}{p}} [g(t)]^{\frac{1}{q}}\right]\right)^{\frac{1}{p}} \geq M^{\frac{-1}{pq}} \left[I_\lambda[f(t)]\right]^{\frac{1}{p}}. \tag{15}$$

Notice that $m\, g(\tau) \leq f(\tau)$, $\tau \in [0, t]$, $t > 0$. It follows that

$$[f(\tau)]^{\frac{1}{p}} \geq m^{\frac{1}{p}} [g(\tau)]^{\frac{1}{p}}. \tag{16}$$

Multiplying the equation (16) by $[g(\tau)]^{\frac{1}{q}}$, we arrive at

$$[f(\tau)]^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} \geq m^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} [g(\tau)]^{\frac{1}{p}} = m^{\frac{1}{p}} [g(\tau)] \tag{17}$$

Multiplying both sides of (17) by $\lambda(\tau, t)$ and integrating respect to τ over (a, t), we obtain

$$I_\lambda \left[[f(t)]^{\frac{1}{p}} [g(t)]^{\frac{1}{q}}\right] \geq m^{\frac{1}{p}} \left[I_\lambda[g(t)]\right]. \tag{18}$$

Hence we have

$$\left(I_\lambda \left[[f(t)]^{\frac{1}{p}} [g(t)]^{\frac{1}{q}}\right]\right)^{\frac{1}{q}} \geq m^{\frac{1}{pq}} \left[I_\lambda[g(t)]\right]^{\frac{1}{q}}. \tag{19}$$

Multiplying the equation (15) and (19), we can draw the desired conclusion easily. □

Also, replacing $f(\tau)$ and $g(\tau)$ by $f(\tau)^p$ and $g(\tau)^q$, $\tau \in [a, t]$, $a \geq 0$ in Theorem 4.1, we obtain the following weighted Hölder's reverse fractional integral inequality:

Corollary 4.1. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in \Delta$ and f and g be two positive function on $[0, \infty[$, such that for all $t > a$, $I_\lambda[f^p(t)] < \infty$, $I_\lambda[g^q(t)] < \infty$. If $0 < m \leq \frac{f(\tau)^p}{g(\tau)^q} \leq M < \infty$, $\tau \in [a, t]$. Then*

$$\left[I_\lambda[f^p(t)]\right]^{\frac{1}{p}} \left[I_\lambda[g^q(t)]\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left[I_\lambda[f(t)g(t)]\right].$$

5. Some other weighted integral inequalities

Now, some integral inequalities of arithmetic and geometric means are proved.

Theorem 5.1. *Let $p \geq 1$, $\lambda \in \Delta$ and let f, g be two positive functions on $[0,+\infty)$ such that for all $t > 0$, $I_\lambda[f^p(t)] < \infty$, $I_\lambda[g^p(t)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [a,t]$ $(a \geq 0)$, then*

$$\left(\frac{(M+1)(m+1)}{M} - 2\right)(I_\lambda[f^p(t)])^{1/p}(I_\lambda[g^p(t)])^{1/p} \leq \left(I_\lambda[f^p(t)]\right)^{2/p} + (I_\lambda[g^p(t)])^{2/p}. \tag{20}$$

Proof. Multiplying inequalities (5) and (7), we get

$$\frac{(M+1)(m+1)}{M}(I_\lambda[f^p(t)])^{1/p}(I_\lambda[g^p(t)])^{1/p} \leq I_\lambda[(f+g)^p(t)])^{2/p}, \tag{21}$$

Besides, applying Minkowski inequality to the right hand side of the last inequality, we get

$$I_\lambda[(f+g)^p(t)])^{2/p} \leq \left((I_\lambda[f^p(t)])^{1/p} + (I_\lambda[g^p(t)])^{1/p}\right)^2. \tag{22}$$

Then, by (21) and (22), with a straightforward calculation follows (20). □

Remark 5.1. *Theorems 3.1 and 5.1 become to Theorem 3.1 and 3.2 of* [4] *in virtue of remark 2.*

Theorem 5.2. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in \Delta$ and f and g be two integrable functions on $[0,\infty]$ such that $0 < m < \frac{f(\tau)}{g(\tau)} < M, \tau \in [a,t]$. Then*

$$I_\lambda[fg(t)] \leq \frac{2^{p-1}M^p}{p(M+1)^p}\left(I_\lambda[f^p + g^p](t)\right) + \frac{2^{q-1}}{q(m+1)^q}\left(I_\lambda[f^q + g^q](t)\right), \tag{23}$$

Proof. Since, $\frac{f(\tau)}{g(\tau)} < M, \tau \in (a,t), a \geq 0$, we have

$$(M+1)f(\tau) \leq M(f+g)(\tau). \tag{24}$$

Taking p^{th} power on both side, multiplying resulting identity by $\lambda(\tau,t)$ and integrating respect τ over (a,t), we get

$$I_\lambda[f^p(t)] \leq \frac{M^p}{(M+1)^p} I_\lambda[(f+g)^p(t)]. \tag{25}$$

On other hand, $0 < m < \frac{f(\tau)}{g(\tau)}, \tau \in (a,t)$, we can write

$$(m+1)g(\tau) \leq (f+g)(\tau), \tag{26}$$

Again, multiplying equation (26) by $\lambda(\tau,t)$ and integrating respect τ over (a,t), we get

$$I_\lambda[g^q(t)] \leq \frac{1}{(m+1)^q} I_\lambda[(f+g)^q(t)]. \tag{27}$$

Now, using Young inequality

$$[f(\tau)g(\tau)] \leq \frac{f^p(\tau)}{p} + \frac{g^q(\tau)}{q}. \tag{28}$$

Multiplying both side of (28) by $\lambda(\tau,t)$ and integrating respect τ over (a,t), we get

$$I_\lambda[f(t)g(t))] \leq \frac{1}{p}\, I_\lambda[f^p(t)] + \frac{1}{q}\, I_\lambda[g^q(t)], \tag{29}$$

from equation (25), (27) and (29) we get

$$I_\lambda[f(t)g(t))] \leq \frac{M^p}{p(M+1)^p}\, I_\lambda[(f+g)^p(t)] + \frac{1}{q(m+1)^q}\, I_\lambda[(f+g)^q(t)], \tag{30}$$

now using the inequality $(a+b)^r \le 2^{r-1}(a^r+b^r), r>1, a,b\ge 0$, we have

$$I_\lambda[(f+g)^p(t)] \le 2^{p-1} I_\lambda[(f^p+g^p)(t)], \tag{31}$$

and

$$I_\lambda[(f+g)^q(t)] \le 2^{q-1} I_\lambda[(f^q+g^q)(t)]. \tag{32}$$

Injecting (31), (32) in (30) we get required inequality (23). □

6. Applications and further results

The following result is on Clarkson's type inequality. He established some inequalities for proving the uniform convexity of L_p and l_p spaces with $1 < p < +\infty$ (see [6]). And, many specialist have used their results in several branches of mathematics, engeniery, etc (see e.g. [2, 7]). This statement is established using the weighted Minkowski's reverse fractional integral inequalities.

Theorem 6.1. *Let $p \ge 1$, $\lambda \in \Delta$ and let f, g be two positive functions on $[0,+\infty)$ such that for all $t>0$, $I_\lambda[f^p(t)] < \infty$, $I_\lambda[g^p(t)] < \infty$. If $0 < 1 < m \le \frac{f(\tau)}{g(\tau)} \le M$, $\tau \in [a,t]$ $(a \ge 0)$, then*

$$I_\lambda[f^p(t)] + I_\lambda[g^p(t)] \le C_{M,m} I_\lambda[(f+g)^p(t)] + C_m I_\lambda[(f-g)^p(t)]. \tag{33}$$

where $C_{M,m} = \frac{M^p(m+1)^p+(M+1)^p}{2(M+1)^p(m+1)^p}$ and $C_m = \frac{1+m^p}{2(m-1)^p}$.

Proof. By (5) and (7), we get

$$I_\lambda[f^p(t)] + I_\lambda[g^p(t)] \le \left(\frac{1}{(m+1)^p} + \frac{M^p}{(M+1)^p}\right) I_\lambda[(f+g)^p(t)]. \tag{34}$$

Besides, by (9) and (10), we have for $c=1$

$$I_\lambda[f^p(t)] + I_\lambda[g^p(t)] \le \left(\frac{1}{(m-1)^p} + \frac{m^p}{(m-1)^p}\right) I_\lambda[(f-g)^p(t)]. \tag{35}$$

Thus, the desired inequality (33) follows by (34) and (35). □

Now, another application on a weighted Randon's reverse integral inequality. Here, we use the Holder's reverse fractional integral inequality established in Theorem 4.1.

Theorem 6.2. *Let $\lambda \in \Delta$ and let $f(x)$ and $g(x)$ be positive and continuous functions. If $n>0$ and $0 < m \le \left(\frac{f(\tau)}{g(\tau)}\right)^{n+1} \le M$, $\tau \in [a,t]$, then*

$$\int_a^t \frac{f^{n+1}(x)}{g^n(x)}\lambda(x,t)dx \le \left(\frac{M}{m}\right)^{n/(n+1)} \frac{\left(\int_a^t f(x)\lambda(x,t)dx\right)^{n+1}}{\left(\int_a^t g(x)\lambda(x,t)dx\right)^{n}}, \quad a<t. \tag{36}$$

Proof. By the condition $0 < m \leq \left(\frac{f(\tau)}{g(\tau)}\right)^{n+1} \leq M$, $\tau \in [a,t]$, $p = n+1$, $q = (n+1)/n$, taking $u(x) = \frac{f(x)}{[g(x)]^{n/(n+1)}}$ and $v(x) = [g(x)]^{n/(n+1)}$ and corollary 4, we obtain

$$\left(\int_a^t \frac{f^{n+1}(x)}{g^n(x)}\lambda(x,t)dx\right)^{1/(n+1)} \left(\int_a^t g(x)\lambda(x,t)dx\right)^{n/(n+1)} \leq \left(\frac{M}{m}\right)^{n/(n+1)^2} \int_a^t f(x)\lambda(x,t)dx$$

and the inequality (36) follows by traightforward calculation in the above inequality. $\square$

Some interesting examples shall be shown for looking the many relations that we could find just considering some special functions and weights. For this reason, we consider the following inequality in the below two examples:

$$\frac{t}{1+t} \leq 1 - e^{-t} \leq \frac{4}{3}\frac{t}{1+t}, \quad 0 \leq t \leq +\infty.$$

Example 6.1. Setting $\lambda(\tau,t) = e^{-\tau}$ on $(0,\infty)$ we get

$$\int_0^{+\infty} (1-e^{-\tau})^p e^{-\tau} d\tau < +\infty \quad \text{and} \quad \int_0^{+\infty} \left(\frac{t}{1+t}\right)^p e^{-\tau} d\tau < +\infty.$$

Then, by Theorem 5.1

$$\frac{3}{2}(I_{e^{-x}}[(1-e^{-x})^p])^{1/p}\left(I_{e^{-x}}\left[\left(\frac{x}{1+x}\right)^p\right]\right)^{1/p} \leq \left(I_{e^{-x}}[(1-e^{-x})^p]\right)^{2/p} + \left(I_{e^{-x}}\left[\left(\frac{x}{1+x}\right)^p\right]\right)^{2/p}.$$

Example 6.2. Also, we can consider $\lambda(\tau,t) = (1+\tau)^{\alpha-1}$ where $\alpha < 0$ and $p = 1$ for getting

$$\int_0^{+\infty} (1-e^{-\tau})(1+\tau)^{\alpha-1} d\tau < +\infty \quad \text{and} \quad \int_0^{+\infty} \left(\frac{t}{1+t}\right)(1+\tau)^{\alpha-1} d\tau < +\infty.$$

Thus, by Theorem 3.1

$$I_{(1+x)^{\alpha-1}}[1-e^{-x}] + I_{(1+x)^{\alpha-1}}\left(\frac{x}{1+x}\right) \leq \frac{5}{7} I_{(1+x)^{\alpha-1}}\left(1 - e^{-x} + \frac{x}{1+x}\right).$$

Moreover, if we consider some particular p, it is possible to get sharp inequalities and bounds.

Example 6.3. If we consider the recent inequalities found by Qi and Mahmoud in [20, Theorem 1], we have

$$\frac{\tan\left(\frac{\pi}{4}x\right)}{\alpha x} \leq \Gamma(x+1) < \frac{\tan\left(\frac{\pi}{4}x\right)}{\beta x}, \quad 0 < x \leq 1,$$

where Γ is the gamma function and the constants $\alpha = 1$ and $\beta = \pi/4$ are the best possible. Thus, for $\lambda(t,x) = \frac{x^2}{(\Gamma(x+1))^2}$ on $[0,1]$ we obtain

$$\int_0^1 \frac{\tan^2\left(\frac{\pi}{4}x\right)}{(\Gamma(x+1))^2} dx < +\infty.$$

Hence, by Theorem 4.1 for $p=q=2$ we get

$$\frac{1}{\sqrt{3}}\left(\int_0^1 \frac{\tan^2\left(\frac{\pi}{4}x\right)}{(\Gamma(x+1))^2}dx\right)^{1/2} \leq \left(\frac{4}{\pi}\right)^{1/4}\int_0^1 \frac{x\tan\left(\frac{\pi}{4}x\right)}{\Gamma(x+1)}dx < +\infty.$$

Example 6.4. Also, by Theorem 3 in [20], we have for any constant τ

$$\mu\exp\left(\frac{x^2}{6-x^2}\right) \leq \Gamma(x+1) \leq \lambda\exp\left(\frac{x^2}{6-x^2}\right), \quad 0\leq x\leq\tau<\sqrt{6},$$

where the constans $\lambda=1$ and $\mu=\Gamma(\tau+1)\exp\left(\frac{\tau^2}{\tau^2-6}\right)$ are the best possible. Besides, setting $\lambda(x,\tau)=\exp\big\{-\frac{x^2}{6-x^2}\big\}/\Gamma(x+1)$ for $0\leq x\leq\tau$, we get by Theorem 6.2

$$\int_0^\tau \frac{\Gamma^n(x+1)}{\big(\exp\big\{\frac{x^2}{6-x^2}\big\}\big)^{n+1}}dx \leq \frac{1}{\mu^n}\frac{\left(\int_0^\tau \exp\big\{-\frac{x^2}{6-x^2}\big\}dx\right)^{n+1}}{\left(\int_0^\tau \frac{dx}{\Gamma(x+1)}\right)^n},$$

where $n>0$ and $0\leq x\leq\tau<\sqrt{6}$.

7. Conclusion

Several results on fractional integral inequalities have been obtained using the classical fractional integro-differentiation operators. Nevertheless, we obtain in this paper, general and exhaustive results on this kind of inequalities that becomes in many well-known results just under the consideration of some particular and simples weights. This paper could lead a specialist to think about the power to consider suitable weighted classes of functions to define good enough operators for getting more fruitful results.

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